Entropy Identity and Material-Independent Equilibrium Conditions in Relativistic Thermodynamics

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Abstract On the basis of the balance equations for energy-momentum, spin, particle and entropy density, an approach is considered which represents a comparatively general framework for special- and general-relativistic continuum thermodynamics. In the first part of the paper, a general entropy density 4-vector, containing particle, energy-momentum, and spin density contributions, is introduced which makes it possible, firstly, to judge special assumptions for the entropy density 4-vector made by other authors with respect to their generality and validity and, secondly, to determine entropy supply and entropy production. Using this entropy density 4-vector, in the second part, material-independent equilibrium conditions are discussed. While in literature, at least if one works in the theory of irreversible thermodynamics assuming a Riemann space-time structure, generally thermodynamic equilibrium is determined by introducing a variety of conditions by hand, the present approach proceeds as follows: For a comparatively wide class of space-time geometries the necessary equilibrium conditions of vanishing entropy supply and entropy production are exploited and, afterwards, supplementary conditions are assumed which are motivated by the requirement that thermodynamic equilibrium quantities have to be determined uniquely.

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1. Introduction

Relativistic thermodynamics for an 1-component material starts out with the balance equations of the particle flux 4-vector N^k, the energy-momentum tensor T^{ik} and the spin tensor $S_{ji}^{\cdot\cdot k}$

$$N_{\cdot;k}^{k} \; = \; 0, \qquad T_{\cdot\cdot;k}^{ki} \; = \; G^i + K^i, \qquad S_{ji;k}^{\cdot\cdot k} \; = \; H_{[ji]} + L_{[ji]} \;^1 \qquad (1)$$

and with the balance of the 4-entropy S^k

$$S_{\cdot;k}^{k} \; = \; \varphi + \sigma, \qquad \sigma \geq 0. \qquad (2)$$

As usual, the semicolon ";" denotes the covariant derivative, T^{ik} is the energy momentum-tensor of a material which is not necessarily symmetric with vanishing covariant derivative, the spin tensor $S_{ji}^{\cdot\cdot k}$ is skew-symmetric in the lower indices, and φ and σ are the entropy supply and the entropy production, respectively. The force K^i and the angular momentum $L_{[ji]}$ are the external sources of the energy momentum tensor and of the spin tensor.

The supply terms φ, K^i, and $L_{[ji]}$ are substitutes for cases in which the energy-momentum tensor and/or the spin tensor do not include all fields, so that additional fields come into account by external sources. Of course, in a field theory describing systems completely by the equations of the fundamental fields, external sources do not occur. If one is forced to introduce supply terms, this shows that the theory is not field-theoretically complete. To complete it, one has to describe the supply terms by additional fundamental fields in such a way that they can be absorbed by the other expressions in the balances (1) [1, 2]. From a thermodynamical point of view, this procedure to include the supplies into effective tensors from the very beginning, is disadvantageous, as we will see below. Here, we imply supply terms for the following reason: Often one considers a situation in which an approximate or a phenomenological description is sufficient, and one does not need a complete description of the system. For example, one need not to imply Maxwell's equations (or the corresponding Lagrangian), if one only intends to regard the influence of a given external electromagnetic field on a charged fluid. This influence can be regarded by assuming that K^i is given a by the Lorentz (volume) force.

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[1]Square brackets are also used to emphasize that the tensor is antisymmetric, $L_{(ji)} = 0$, especially for H_{ji} and L_{ji}

The G^i and $H_{[ij]}$ are internal source terms caused by the choice of a special space-time and by the spin-momentum-energy coupling (SMEC). For instance in Einstein-Cartan geometry, the G^i and $H_{[ij]}$ are caused by the torsion and depend as coupling terms on the energy-momemtum and on the spin tensor. We call a theory for which the G^i and $H_{[ij]}$ vanish identically SMEC-free.

In contrast to Special Relativity Theory (SRT) and Einstein-Cartan Theory (ECT), General Relativity Theory (GRT), makes no general statements on the structure of spin and spin balances, except for that here does not occur a spin tensor as explicit source of gravity. The spin of the matter source has only an implicit influence on the gravitational field insofar, as the source term in Einstein's equations (the symmetric metrical energy-momentum tensor), differs for different kinds of spinorial matter. In some cases, where the total set of equations consists of Einstein's equations coupled to field equations of phenomenological matter, one can derive from this set, beside the energy-momentum balance, also a spin balance. For a Weyssenhoff fluid, particularly follows beside $K^i = 0$ the SMEC-term $H_{[ik]} = T_{[ik]}$ [3].

The non-negative entropy production σ in (2) represents the strong formulation of the Second Law of thermodynamics in field theories. The inequality

$$S^k_{:;k} - \varphi \ \geq \ 0 \tag{3}$$

is called the dissipation inequality.

The relations (1) and (2) are the relativistic generalization of the balance equations of non-relativistic continuum thermodynamics. In their special-relativistic version – with vanishing spin tensor, vanishing supply terms and vanishing SMEC-terms – they were introduced by Eckart [4] and Kluitenberg [5]. The quantities appearing in (1) and (2) are tensors with respect to Lorentz transformations, and the derivatives denoted by the semicolon have to be read as partial derivatives, if one refers to inertial systems, and as covariant derivatives (with the Christoffel symbols as components of the Levi-Civita connection), if non-inertial reference systems are considered.

Assuming that dynamics takes place in a curved space-time, the equations (1) and (2) describing this dynamics have to be interpreted as generally covariant tensor equations in this chosen space-time. This means, that the basic quantities in (1) and (2) must be considered as tensors with respect to

arbitrary systems of reference (observers) and that the semicolon derivative is the covariant derivative defined by the space-time geometry belonging to the theory of gravitation under consideration.

In case of GRT, a Riemann space-time is assumed whose covariant derivative is given by the Levi-Civita connection. (The above-mentioned Minkowski space-time of special relativity theory is a Riemann space-time with vanishing curvature, i.e., the case for which gravitation curving the space-time is neglected.) In generalizations of GRT, instead of the Riemann space-time defined by the metric as a primary quantity, geometrically more rich geometries are considered. For instance, one can introduce space-times that are characterized by additional quantities like torsion and non-metricity which are independent of the metric [6]. In these space-times, non-vanishing SMEC-terms G^i and $H_{[ij]}$ appear which are describing the coupling of the torsion to spin and energy-momentum.

As long as one does not consider the gravitational field equations specifying the space-time, but only a given curved space-time of one of the above mentioned types is assumed, one can work with the general framework given by (1) and (2). However, the situation changes drastically, if completing the theory by incorporating the gravitational equations. In GRT, the gravitational field and thus the curvature of the Riemann space-time is determined by Einstein's equations

$$R_{ik} - \frac{1}{2}g_{ik}R \;=\; -\kappa\hat{T}_{ik}, \tag{4}$$

and therefore one has to assume a symmetric energy-momentum tensor $\hat{T}_{ik}$ and vanishing K^i (G^i and $H_{[ik]}$ vanish in GRT, if there is no spin. In this case GRT is SMEC-free)

$$\hat{T}^{ik} \;=\; \hat{T}^{ki}, \qquad \hat{T}^{ki}_{..;k} \;=\; 0. \tag{5}$$

The condition $(5)_1$ expresses the fact that in GRT way is given for matter of non-vanishing spin, but that the spin of matter is only insofar a source of gravitation, as it can be reflected by terms contained in the symmetric energy-momentum tensor $\hat{T}^{ik}$. If one considers matter with spin having a non-symmetric energy-momentum tensor of matter, this matter can be incorporated into GRT by symmetrizing the energy-momentum tensor, such that it satisfies (5). As to condition $(5)_2$, it requires in $(1)_2$ a supply term

4

K^i, of the form $K^i = \Theta^{ik}_{\cdot\cdot;k}$, such that it can be rewritten into a divergence of a second-rank tensor $\hat{T}^{ik} - \Theta^{ik}$. In some cases it was shown that, starting out with conditions (1), (5) can also be established (see e.g. [1]).

For generalizations of GRT like Einstein-Cartan theories (see e.g. [7, 8, 9]), due to the changed geometry, one finds generalized Bianchi identities and gravitational equations modifying Einstein equations (4). Also in these cases, one has to arrange that these new identities and equations are compatible with the balance equations. Thus, again, one obtains restrictions to the balances or to the material field equations.

In this paper, we do not assume a special theory of gravitation, but we discuss the balance equations (1) and (2) for the general case of a curved space-time with a given background specifying metric and connection. Attention is concentrated on the analysis of the dissipation inequality (2) from the point of view that it has to be satisfied by any ansatz for the entropy vector. The results are generally valid in SRT and have to be specified by additional conditions, especially for G^i and $H_{[ij]}$, in the case of non-SMEC-free relativistic gravitational theories.

For solving the system of differential equations (1) and (2) in different chosen geometries according to (4) or according to other gravitational theories, constitutive equations are needed, because the balances and field equations are valid for arbitrary, for the present unspecified materials. Here N^k, T^{ik}, S^k and $S^{\cdot\cdot l}_{ik}$ are constitutive mappings defined on a large state space (no after-effects) [10]

$$\mathbf{z} = (g_{ik}, \mathcal{T}^{\cdot\cdot l}_{ik}, n, e, s_{ik}\Xi_k, u^k, ...), \tag{6}$$

which may contain the geometrical fields, such as the metric g_{ik}, the torsion $\mathcal{T}^{\cdot\cdot l}_{ik}$, and the wanted basic fields $(n, e, s_{ik}, \Xi_k, u^k)$ (particle number density, energy density, spin density, spin density vector and an other for the present arbitrary time-like vector field u^k) and beyond them other fields ..... which depend on the considered material and which are of no interest here, because we are looking for *material-independent* properties. Consequently, a special constitutive equation will not appear in this paper.

In relativistic irreversible thermodynamics, stable thermodynamical equilibria are characterized by the fact, that the temperature 4-vector is a Killing vector [11]. But this is only true, if T^{ik} is symmetric and if the space-time is SMEC-free, properties which are not valid in general and which are not

5

presupposed here. Therefore the question arises and will be answered: Are there equilibrium conditions independently of constitutive properties in the framework of a general gravitation theory?

The paper is organized as follows: starting out with the 3-1-split of the quantities appearing in (1) and (2), we derive an identity for the entropy 4-vector [12] in Sect.3. Using Eckart's interpretation of the time-like vector u^k as 4-velocity of the material [4] in Sect.4, we are formulating material-independent equilibrium conditions in Sect.6. Proofs can be found in the appendices.

2 . 3-1-Split

The normalized time-like vector field u^k included in the state space (6) (signature of the metric is –2)

$$u^k u_k \;=\; a^2 \;>\; 0 \quad \longrightarrow \quad u^k u_{k;m} \;=\; 0 \tag{7}$$

is first of all arbitrary and can therefore be chosen in different ways. Here, it is introduced for spitting the quantities into their parts parallel and perpendicular to u^k. This split allows for a special interpretation later on. By introducing the projector belonging to (7)

$$h^k_l \;:=\; g^k_l \;-\; \frac{1}{a^2} u^k u_l, \tag{8}$$

we obtain as shown in appendix 1

$$N^k \;=\; \frac{1}{a^2} n u^k \;+\; n^k, \qquad S^k \;=\; \frac{1}{a^2} s u^k \;+\; s^k, \tag{9}$$

$$T^{ik} \;=\; \frac{1}{a^4} e u^i u^k \;+\; \frac{1}{a^2} u^i p^k \;+\; \frac{1}{a^2} q^i u^k \;+\; t^{ik}, \tag{10}$$

$$S^{··l}_{ik} \;=\; \left(\frac{1}{a^2} s_{ik} + \frac{1}{a^4} u_{[i} \Xi_{k]} \right) u^l + s^{··l}_{ik} + \frac{1}{a^2} u_{[i} \Xi^{·l}_{k]}. \tag{11}$$

Here the following abreviations are introduced

$$n \;:=\; N^k u_k, \quad n^k \;:=\; h^k_l N^l, \tag{12}$$

$$s \;:=\; S^k u_k, \quad s^k \;:=\; h^k_l S^l, \tag{13}$$

$$e := u_l u_m T^{lm}, \; p^k := h^k_l u_m T^{ml}, \quad q^k \;:=\; h^k_l u_m T^{lm}, \; t^{ik} := h^i_l h^k_m T^{lm} \tag{14}$$

$$s_{ik} := S^{··c}_{ab} h^a_i h^b_k u_c, \quad \Xi_k \;:=\; 2 S^{··c}_{ab} u_c u^a h^b_k, \tag{15}$$

$$s^{··l}_{ik} := S^{··c}_{ab} h^a_i h^b_k h^l_c, \quad \Xi^{·l}_k \;:=\; 2 S^{··c}_{ab} h^l_c u^a h^b_k. \tag{16}$$

The physical interpretation of the quantities in (12) to (16) remains uncertain, as long as the the time-like vector field u^k is not interpreted. Later on, the quantities in (15) and (16) are recognized as follows [13]: s_{ik} is the spin density, $s_{ik}^{\cdot\cdot l}$ the couple stress, Ξ_k the spin density vector and Ξ_k^l is the spin stress. These quantities are not independent of each other, but they are coupled by the spin axioms [13] which we will use later. Independently of any interpretation, the 4-entropy satisfies an identity which is derived in the next section.

3. The Entropy Identity

In literature, one finds different approaches to a special- and general-relativistic conception of entropy. Most of them is in common that entropy is described by a 4-vector, but there are proposed different expressions for it (which generally do not incorporate spin terms). For instance, in [14] is generalized the non-relativistic expression of the internal energy U for constant temperature and composition

$$U \;=\; TS - \mu n \tag{17}$$

(T = rest temperature, S = entropy, μ = chemical potential) which results by differentiation in classical thermodynamics of discrete systems together with the Gibbs equation in the Gibbs-Duhem equation. This yields an entropy 4-vector

$$S^k \;=\; \mu N^k + \frac{u_m}{T}T^{km} + p\frac{u^k}{T} \;=\; \mu N^k + \frac{u_m}{T}\left[T^{km} + p\delta^{km}\right] \tag{18}$$

(p = pressure).

Other authors make an ansatz for the entropy vector such that its covariant divergence becomes a relativistic generalization of the Carnot-Clausius relation,

$$d_e S \;=\; \frac{\delta Q}{T}. \tag{19}$$

Here "d_e" denotes a change caused by an external supply (see e.g. [11]). Accordingly, they assume[2]

$$S^k \;=\; \mu N^k + \frac{u_m}{T}T^{km}. \tag{20}$$

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[2]For the present, a vector O_m is introduced which later on is identified to be equal to u_m/T.

The procedure in this paper is quite different: We do not make ansatzes of the entropy vector S^k, but we start out with an identity which runs as follows:

$\diamond$ Independently of the special interpretation of the time-like vector field u^k, the following identity for the 4-entropy is valid:

$$S^k \equiv (s^k - \lambda q^k - \mu n^k - \Lambda^m \Xi_m^{\cdot k}) + (\mu N^k + \xi_l T^{kl} + \zeta^{nm} S_{nm}^{\cdot \cdot k}), \qquad (21)$$

with the following abbreviations:

$$\lambda \text{ arbitrary scalar, } \Lambda^k \text{ arbitrary tensor field of 1st order,} \qquad (22)$$

$$\mu := \frac{1}{n}(s - \lambda e - \Lambda^m \Xi_m), \quad \xi_l := \lambda u_l, \quad \zeta^{nm} := 2u^n \Lambda^p h_p^m. \qquad \diamond \quad (23)$$

$\square$ The proof is easy: Starting out with the relations (9)

$$s^k = S^k - \frac{s}{n}\left(N^k - n^k\right), \qquad (24)$$

we obtain from $(14)_3$, (8), $(14)_1$ and $(9)_1$

$$q^k = u_m T^{km} - \frac{1}{a^2} e u^k = u_m T^{km} - \frac{e}{n}\left(N^k - n^k\right). \qquad (25)$$

From $(16)_2$ follows by use of (8), $(15)_2$ and $(9)_1$

$$\Xi_m^{\cdot k} = 2u^p h_m^q S_{pq}^{\cdot \cdot k} - \frac{2}{a^2} u^p h_m^q S_{pq}^{\cdot \cdot r} u^k u_r =$$
$$= 2u^p h_m^q S_{pq}^{\cdot \cdot k} - \frac{1}{n}\Xi_m\left(N^k - n^k\right). \qquad (26)$$

Summing up the last three equations multiplied with λ and Λ^m, we obtain

$$s^k - \lambda q^k - \Lambda^m \Xi_m^{\cdot k} =$$
$$= S^k - \lambda u_m T^{km} - 2\Lambda^m u^p h_m^q S_{pq}^{\cdot \cdot k} + \frac{1}{n}\left(-s + \lambda e + \Lambda^m \Xi_m\right)\left(N^k - n^k\right) \quad (27)$$

which is identical to (21). $\qquad\qquad \square$

Consequently, the identity (21) is valid for arbitrary λ and Λ^m and for all time-like vector fields u^k.

The ad-hoc chosen entropy vector (20) is in accordance with the identity (21) by setting

$$\Lambda^m := 0, \quad \zeta^{nm} := 0, \quad \lambda := \frac{1}{T}, \quad s^k := \lambda q^k + \mu n^k. \qquad (28)$$

But it is not quite clear, if (20) represents the most general ansatz also without spin, since the identity (21) allows for adding a space-like vector, the first bracket in (21). To clarify this question and for incorporating spin, we do not start out with a specific ansatz for the entropy vector, but with the identity (21) in Sect.5.

In contrast to the expression (20) for the entropy 4-vector, (18) is not in accordance with the identity (21). If the chemical potential μ and the energy-momentum tensor T^{km} in (18) are the same quantities as in (21), we obtain for the spin-free case by comparing (18) with (21) the false equation

$$p\frac{u^k}{T} \stackrel{f!}{=} s^k - \lambda q^k - \mu n^k. \qquad (29)$$

Consequently, μ and T^{km} in (18) are different from those in (21), or (18) is wrong.

Without any restriction of generality, from (21), (23), (15)$_2$ and (16)$_2$ follows, that Λ^m can be chosen orthogonal to u^m

$$\Lambda^m \doteq \Lambda^p h_p^m. \qquad (30)$$

Later on, this choice makes an interpretation of Λ^m more easy.

In the next section, we will identify the time-like u^k field, thus resulting in an interpretation of the quantities (12) to (16).

4. Eckart and Landau-Lifshitz Interpretation

Two different interpretations of the u^k can be found in literature: the first one is due to Landau-Lifshitz [15], the second one due to Eckart [4].

Landau-Lifshitz choose u^k as an eigenvector of the energy-momentum tensor

$$u_m T^{km} = \frac{e}{a^2} u^k. \qquad (31)$$

By (25), this choice results in

$$q^k \equiv 0. \tag{32}$$

That means, this choice fixes 3 of the 16 free components of the energy-momentum tensor. Because this tensor represents a constitutive mapping, (31) is introducing a special constitutive property. Because we are looking for material independent statements, we do not accept the Landau-Lifshitz choice (31) of u^k.

Eckart's choice of u^k along $(9)_1$

$$u^k := \frac{a^2}{n} N^k, \quad a \equiv c, \quad \text{or} \quad n^k \equiv 0, \tag{33}$$

is more general than (31): It does not restrict the energy-momentum tensor or the spin tensor, because N^k is not a part of T^{ik} or $S^{..l}_{ik}$. A second advantage is its illustrative interpretation: because the particle flux is purely convective and has no conductive part, u^k is according to (33) the material 4-velocity, and we obtain for the particle number flux according to Eckart

$$N^k = \frac{1}{c^2} n u^k, \tag{34}$$

an expression which is widely accepted in relativistic continuum physics.

5. Entropy Balance

We now introduce Eckart's version into the entropy identity (21)

$$S^k \equiv (s^k - \lambda q^k - \Lambda^m \Xi^{\cdot k}_m) + (\mu N^k + \xi_l T^{kl} + \zeta^{nm} S^{..k}_{nm}), \tag{35}$$

and (22) and (23) are still valid.

In order to determine the entropy vector in accordance with this identity, one can exploit the entropy balance (2)

$$S^k_{;k} = \varphi + \sigma, \tag{36}$$

and by differentiating (35) and by use of the balance equations (1), we obtain

$$S^k_{;k} = (s^k - \lambda q^k - \Lambda^m \Xi^{\cdot k}_m)_{;k} +$$
$$+\mu_{,k} N^k + \xi_{l;k} T^{kl} + \zeta^{nm}_{;k} S^{..k}_{nm} + \xi_l [G^l + K^l] + \zeta^{nm} (H_{[nm]} + L_{[nm]}). \tag{37}$$

10

To interpret this balance by physics, one has to identify the supply and production terms φ and σ. To this end, we refer to classical thermodynamics which defines the entropy supply as the energy supply r times the reciprocal rest temperature

$$\varphi \; := \; \frac{r}{T}. \tag{38}$$

The energy supply itself is caused by the external forces K^i and by the external moments $L_{[ik]}$. Consequently, we have by definition

$$r \; := \; u_i K^i + s_{lm} \Theta^{[lm][ik]} L_{[ik]}. \tag{39}$$

The tensor $\Theta^{[lm][ik]}$ which connects the spin to the external moments does not need to be specified for our purposes. Interesting is that the rest temperature T is introduced by $T = r/\varphi$ according to (38).

The entropy supply can be read off from (37), and a comparison with (38) results in

$$\varphi \; = \; \xi_l K^l + \zeta^{nm} L_{[nm]} \; = \; \frac{1}{T} u_i K^i + \frac{1}{T} s_{lm} \Theta^{[lm][ik]} L_{[ik]}. \tag{40}$$

This enables one to determine λ and Λ^m which were arbitrary up to now. From $(23)_2$ and $(23)_3$ follows

$$\xi_i \; = \; \frac{u_i}{T} \; = \; \lambda u_i \tag{41}$$

$$\zeta^{[ik]} \; = \; \frac{1}{T} s_{lm} \Theta^{[lm][ik]} \; = \; 2 u^{[i} h_m^{k]} \Lambda^m \; = \; u^i h_m^k \Lambda^m - u^k h_m^i \Lambda^m. \tag{42}$$

Multiplication of (42) with u_i and taking (30) into consideration results in

$$\lambda \; = \; \frac{1}{T}, \qquad \Lambda^k \; = \; \frac{1}{c^2} \frac{s_{lm} u_i}{T} \Theta^{[lm][ik]}. \tag{43}$$

The vector (41) which is in accordance with the former definition $(28)_3$ is called the 4-temperature. The vector $(43)_2$ which later on will play a role for formulating the equilibrium conditions of the spin is called the temperature-spin.

After having determined the supply terms according to (41) and (42), the remaining terms on the left-hand side of (37) have to be considered as the

entropy production according to (36)

$$\sigma = (s^k - \frac{1}{T}q^k - \Lambda^m \Xi^{\cdot k}_m)_{;k} + \frac{1}{T}u_l G^l + \frac{2}{T}u^{[i}\Lambda^{k]}H_{[ik]} +$$
$$+ \mu_{,k}N^k + (\frac{1}{T}u_l)_{;k}T^{kl} + 2(u^{[n}\Lambda^{m]})_{;k}S^{\cdot\cdot k}_{nm} \geq 0. \qquad (44)$$

This expression includes three terms of different characters, a divergence term of a space-like vector, the SMEC-terms and terms according to the usual flux-force scheme [16] of the entropy production. The divergence term contains fluxes which does not contribute to the entropy production. Therefore we define the entropy flux by

$$s^k := \frac{1}{T}q^k + \Lambda^m \Xi^{\cdot k}_m. \qquad (45)$$

Finally taking (45) into account, the entropy production (44) results in

$$\sigma = \frac{1}{T}u_l G^l + \frac{2}{T}u^{[i}\Lambda^{k]}H_{[ik]} + \mu_{,k}N^k + (\frac{1}{T}u_l)_{;k}T^{kl} + 2(u^{[n}\Lambda^{m]})_{;k}S^{\cdot\cdot k}_{nm} \geq 0. \quad (46)$$

The entropy follows from (35), (41) and (42)

$$S^k = \mu N^k + \frac{1}{T}u_l T^{kl} + 2u^{[n}\Lambda^{m]}S^{\cdot\cdot k}_{nm}. \qquad (47)$$

In contrast to the entropy production, the entropy does not contain SMEC-terms which are generated by differentiation. For vanishing spin density, (47) coincides with the ansatz (20). But (47) is a derived relation and not only a guessed ansatz. Beyond that, it includes the spin, and also the entropy flux density (45) and the entropy production density (46) follow consistently by the same procedure including the spin.

In [11] the possibility is briefly discussed, if the ansatz (20) for the entropy can be extended by adding a time-like vector. This possibility is excluded by the identity (35) which allows to add only a space-like vector, the first bracket in (35).

The entropy (47) has the form of a sum of products which can be written symbolically

$$S^k = \mathbf{X} \circ \mathbf{Y}^k. \qquad (48)$$

The entropy balance equation (36) becomes

$$S^k_{;k} \;=\; \mathbf{X}_{;k} \circ \mathbf{Y}^k + \mathbf{X} \circ \mathbf{Y}^k_{;k} \;=\; \sigma + \varphi. \tag{49}$$

According to (46), we obtain only for space-times of vanishing SMEC-terms, $G^l \equiv 0, \; H_{[ik]} \equiv 0$

$$\sigma \;=\; \mathbf{X}_{;k} \circ \mathbf{Y}^k \quad \rightarrow \quad \varphi \;=\; \mathbf{X} \circ \mathbf{Y}^k_{;k}. \tag{50}$$

According to (46), the entropy production density has not the usual form of a product of "forces" $\mathbf{X}_{;k}$ and "fluxes" $\mathbf{Y}^k$, because the SMEC-terms vanish only for special space-times, but not in general.

6. Equilibrium Conditions

Equilibrium states are defined by equilibrium conditions. We have to distinguish between necessary and supplementary equilibrium conditions. The necessary and the supplementary equilibrium conditions together represent sufficient equilibrium. We will mark both kinds of equilibrium conditions differently: the necessary ones by $\overset{\bullet}{=}$, the supplementary ones by $\overset{\cdot}{=}$. For the present, we consider the necessary conditions in the next section.

6.1 Necessary equilibrium conditions

The necessary equilibrium conditions are given by vanishing entropy production density (46) and vanishing entropy supply density (40)

$$\sigma_{eq} \;\overset{\bullet}{=}\; 0, \quad \varphi_{eq} \;\overset{\bullet}{=}\; 0 \quad \longrightarrow \quad S^{k\ eq}_{;k} \;=\; 0. \tag{51}$$

(equilibrium quantities are marked by $_{eq}$ or by ^{eq} in the sequel) and vanishing entropy flux density

$$s^k_{\ eq} \;\overset{\bullet}{=}\; 0. \tag{52}$$

The implication in (51) follows from (36).

For the present, we will exploit the entropy supply density $(51)_2$ by starting out with (40). Because the force K^i is independent of the momentum $L_{[ik]}$, the part of the necessary equilibrium conditions belonging to the entropy supply splits into two parts and using (42) and (30), we obtain

$$u_i{}^{eq} K^i{}_{eq} \;=\; 0, \quad 2(u^{[i}\Lambda^{k]})_{eq} L_{[ik]}{}^{eq} \;=\; 0. \tag{53}$$

13

From (53) we read off, that for the present neither the external forces nor the external moments have to be zero in equilibrium. Using the balance equations $(1)_{2,3}$, we obtain

$$u_i{}^{eq}[T^{ki}_{;k} - G^i]_{eq} = 0, \qquad (u^{[i}\Lambda^{k]})_{eq}[S^{··j}_{ik}{}_{;j} - H_{[ik]}]_{eq} = 0. \qquad (54)$$

From (47) follows by $(51)_3$

$$0 = (\mu N^k)^{eq}_{;k} + \left(\frac{1}{T}u_l T^{kl}\right)^{eq}_{;k} + 2\left(u^{[n}\Lambda^{m]}S^{··k}_{nm}\right)^{eq}_{;k}. \qquad (55)$$

The N^k, T^{kl} and $S^{··k}_{nm}$ are not independent of each other, because they are coupled by constitutive equations and by the SMEC-terms. Therefore we cannot state that each term of the sum (55) vanishes. The equilibrium condition (55) is only one equation which cannot describe equilibrium completely. Therefore we need supplementary equilibrium conditions beyond (51)and (52). These conditions will be considered in the next section.

6.2 Supplementary equilibrium conditions

6.2.1 Supply conditions

According to the necessary condition $(53)_1$, the power of the external forces is zero in equilibrium. From that one cannot conclude that the external forces vanish themselves in equilibrium. There exist an easy criterion for testing whether the external forces vanish in equilibrium: Starting out again with $(53)_1$, we see that in equilibrium the 4-component of the force is zero in the rest system, marked by ^{R},

$$^R K^4{}_{eq} = 0. \qquad (56)$$

If now also the 3-components of the force vanish in the rest system

$$^R K^\alpha{}_{eq} = 0, \qquad \alpha = 1, 2, 3, \qquad (57)$$

we obtain the very special supplementary equilibrium condition

$$K^i{}_{eq} \doteq 0 \qquad (58)$$

for the external forces.

According to $(43)_2$, the necessary equilibrium condition $(53)_2$, depends on the

14

spin density $(15)_1$ and the temperature. There may be non-zero $\Lambda^k{}_{eq}$-fields depending on the external moments in such a way that $(53)_2$ is satisfied, but this situation is so strange, that we do not take this seriously into consideration. Consequently, we obtain two supplementary equilibrium conditions

$$\Lambda^k{}_{eq} \doteq 0 \qquad \cup \qquad L_{[ik]}{}^{eq} \doteq 0, \tag{59}$$

that means, the external moments have to vanish in equilibrium in systems of non-vanishing spin. If the external moments do not vanish, the system must be spin-free in equilibrium. These statements are true except for the exotic situation that $(53)_2$ is satisfied for non-vanishing $\Lambda^k{}_{eq}$ and $L_{[ik]}{}^{eq}$.

6.2.2 N^k-Condition

To begin with the supplementary equilibrium conditions, we consider by use of $(1)_1$, (34) and the abbreviation $^\bullet := {}_{;k}u^k$

$$\left(\mu N^k\right)_{;k} = \mu_{,k}\frac{1}{c^2}nu^k = \frac{1}{c^2}n\mu^\bullet. \tag{60}$$

Because it is obvious that there are no non-vanishing material time derivatives in equilibrium except that of the acceleration $u^{k\bullet}$ [3], we demand as a first supplementary equilibrium condition

$$\boxplus^\bullet \doteq 0, \quad \boxplus \neq u^k \qquad \longrightarrow \qquad \left(\mu N^k\right)_{;k}{}^{eq} = 0. \tag{61}$$

Consequently, we obtain by (34)

$$0 = \left(\mu N^k\right)_{;k}{}^{eq} = \left(\mu\frac{1}{c^2}nu^k\right)^{eq}_{;k} = \left(\mu\frac{1}{c^2}n\right)^\bullet_{eq} + \mu\frac{1}{c^2}nu^k_{;k}{}^{eq} \tag{62}$$

which by $(61)_1$ results in

$$u^k_{;k}{}^{eq} = 0. \tag{63}$$

Further we obtain by (63) and $(61)_1$

$$\left(\frac{1}{T}u^k\right)_{;k} = \frac{1}{T}u^k_{;k} + \left(\frac{1}{T}\right)^\bullet \qquad \longrightarrow \qquad \left(\frac{u^k}{T}\right)_{;k}{}^{eq} = 0. \tag{64}$$

Hence, the vanishing first term in (55) is exploited by applying the supplementary equilibrium condition (61). Now we will consider the next term.

---

[3]The $^\bullet$ is the relativistic analogue to the non-relativistic material time derivative d/dt which describes the time rates of a rest-observer. Therefore, d/dt is observer-independent and zero in equilibrium [17, 18].

6.2.3 $\mathbf{T}^{kl}$-Condition

Using $(25)_1$, we obtain for the second term in (55)

$$\left(\frac{u_l}{T}T^{kl}\right)_{;k} = \left(\frac{q^k}{T} + \frac{1}{c^2 T}eu^k\right)_{;k} = \left(\frac{q^k}{T}\right)_{;k} + \left(\frac{e}{c^2 T}\right)^{\bullet} + \frac{e}{c^2 T}u^k_{;k} \quad (65)$$

which by use of $(61)_1$ and (63) results in

$$\left(\frac{u_l}{T}T^{kl}\right)^{eq}_{;k} = \left(\frac{q^k}{T}\right)^{eq}_{;k} = \left(\frac{1}{T}\right)^{eq}_{,k}q^k_{eq} + \left(\frac{1}{T}\right)^{eq}q^{k\,eq}_{;k}. \quad (66)$$

The first term of the right-hand side represents the dissipation due to heat conduction which is zero in equilibrium, a statement which represents an other supplementary equilibrium condition

$$\left(\frac{1}{T}\right)^{eq}_{,k}q^k_{eq} \;\dot{=}\; 0 \quad \longrightarrow \quad q^k{}_{eq} = 0. \quad (67)$$

Because there are equilibria with non-vanishing temperature gradient (e.g. in gravitational fields) and because the dissipation due to heat conduction is always not negative

$$\left(\frac{1}{T}\right)_{,k}q^k \;\geq\; 0, \quad (68)$$

and the heat flux density depends continuously on the temperature gradient, the conclusion in (67) is the only possible one [19]. But it is also obvious that there are no heat fluxes in equilibrium. From $(67)_2$ follows

$$q^{k\,eq}_{;k} = 0, \qquad \left(\frac{q^k}{T}\right)^{eq} = 0. \quad (69)$$

Taking $(67)_2$ into account, (66) results by use of $(54)_1$ in

$$0 = \left(\frac{u_l}{T}T^{kl}\right)^{eq}_{;k} = \left(\frac{u_l}{T}\right)^{eq}_{;k}T^{kl}_{eq} + \frac{u_l}{T}^{eq}G^l_{eq} \quad (70)$$

We now consider the special case that the energy-momentum tensor is symmetric in equilibrium (what is not the case in general). Then (70) results in

$$T^{[kl]}_{eq} \equiv 0 \quad \rightarrow \quad (\frac{1}{T}u_{(l);k})^{eq}T^{kl}{}_{eq} = -\frac{u_l}{T}^{eq}G^l_{eq}. \quad (71)$$

16

In general, we cannot conclude from $(71)_2$, that the temperature 4-vector u_l/T is killing in equilibrium even for SMEC-free space-times,

$$(\frac{1}{T}u_l)_{;k}{}^{eq} + (\frac{1}{T}u_k)_{;l}{}^{eq} \stackrel{?}{=} 0, \tag{72}$$

because we do not presuppose a symmetric T^{kl}, as it was assumed in [11]. Presupposing (72), no additional equilibrium conditions would follow for the symmetric T^{kl}, because (71) is satisfied identically for SMEC-free space-times. We now treat the general case.

After a short calculation, we obtain in non-equilibrium by using (10)

$$\left(\frac{u_l}{T}\right)_{;k}T^{kl} = \frac{1}{T}u_l{}^\bullet\frac{1}{c^2}p^l + \frac{1}{T}u_{l;k}t^{kl} + \left(\frac{1}{T}\right)^\bullet\frac{1}{c^2}e + \left(\frac{1}{T}\right)_{,k}q^k. \tag{73}$$

Inserting

$$u_l{}^\bullet p^l = -u_l p^{l\bullet} \tag{74}$$

and taking $(61)_1$ and $(67)_2$ into account, we obtain in equilibrium

$$\left(\frac{u_l}{T}\right)_{;k}{}^{eq}T^{kl}{}_{eq} = \frac{1}{T}u_{l;k}{}^{eq}t^{kl}{}_{eq}, \tag{75}$$

and (70) results in

$$0 = u_{l;k}{}^{eq}t^{kl}{}_{eq} + u_l^{eq}G^l_{eq}. \tag{76}$$

As we can see easily, the following identity is valid

$$0 = u_{l;k}{}^{eq}t^{kl}{}_{eq} + u_l^{eq}G^l_{eq} = u_{l;k}{}^{eq}\left[t^{kl}{}_{eq} + \frac{u_p^{eq}G^p_{eq}}{u_{p;q}{}^{eq}A^{qp}}A^{kl}\right], \tag{77}$$

for all A^{kl} with

$$u_{p;q}{}^{eq}A^{qp} \neq 0, \qquad u_p{}^\bullet{}_{eq}u_q{}^{eq}A^{qp} = 0. \tag{78}$$

We need the second property for later use. Consequently, we can introduce non-unique modified stress tensors which include the SMEC-term

$$\tau^{kl} := t^{kl}{}_{eq} + J^{kl}, \qquad J^{kl} := \frac{u_p^{eq}G^p_{eq}}{u_{p;q}{}^{eq}A^{qp}}A^{kl} \tag{79}$$

17

and (76) results in

$$0 = u_{l;k}{}^{eq}\tau^{kl}, \tag{80}$$

a result which can be also expressed in an other way.

As usual [11], we introduce the kinematical invariants by the following definitions

$$\text{shear:} \qquad \sigma_{ab} := u_{(a;b)} - \frac{1}{c^2}u^{\bullet}{}_{(a}u_{b)} - \frac{1}{3}\Theta h_{ab}, \tag{81}$$

$$\text{rotation:} \qquad \omega_{ab} := u_{[a;b]} - \frac{1}{c^2}u^{\bullet}{}_{[a}u_{b]}, \tag{82}$$

$$\text{acceleration:} \qquad u^{\bullet}{}_a := u_{a;b}u^b, \tag{83}$$

$$\text{expansion:} \qquad \Theta := u^a_{;a}. \tag{84}$$

According to their definitions, we obtain from (8) and from the normalization of the 4-velocity (7) and $(33)_2$

$$h_{ab}u^b = u^{\bullet}{}_a u^a = \sigma_{ab}u^b = \omega_{ab}u^b = 0. \tag{85}$$

From (81) and (82) follows the velocity gradient

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\Theta h_{ab} + \frac{1}{c^2}u^{\bullet}{}_a u_b, \tag{86}$$

and we obtain

$$u_{l;k}\tau^{kl} = [\sigma_{lk} + \omega_{lk}]\tau^{kl} + \frac{1}{3}\Theta h_{lk}\tau^{kl} + \frac{1}{c^2}u^{\bullet}{}_l u_k\tau^{kl}. \tag{87}$$

By use of (63) and $(78)_2$, (80) results in

$$u_{l;k}{}^{eq}\tau^{kl} = [\sigma_{lk} + \omega_{lk}]^{eq}\tau^{kl} = \sigma_{lk}^{eq}\tau^{(kl)} + \omega_{lk}^{eq}\tau^{[kl]} = 0. \tag{88}$$

Because the symmetric and the antisymmetric part of the stress tensor are independent of each other, we can split (88) into

$$\sigma_{lk}^{eq}\tau^{(kl)} = 0, \qquad \omega_{lk}^{eq}\tau^{[kl]} = 0. \tag{89}$$

Because the tensor A^{kl} in $(79)_2$, and consequently also J^{kl} in $(79)_1$, can be chosen arbitrarily, the SMEC-term can be distributed freely on the shear or rotation terms: If A^{kl} is chosen to be symmetric, no part of the SMEC-term appears in the rotation part and vice-versa.

18

The equilibrium conditions (89) can be interpreted differently: If we are looking for equilibrium conditions which are the same for all space-times and materials, that means, they are valid for arbitrary $\tau^{(kl)}$ and $\tau^{[kl]}$, we obtain

$$\sigma_{lk}^{eq} \doteq 0, \qquad \omega_{lk}^{eq} \doteq 0 \tag{90}$$

as supplementary equilibrium conditions.

The second interpretation is as follows: Because (89) are derived material-independently, there may be shear and rotation fields different from zero satisfying (89) for special chosen space-times and materials. That means, there are special material- and space-time-dependent equilibria having non-vanishing shear and/or rotation. By these remarks, the second necessary equilibrium condition (70) is exploited, and we have now to consider the equilibrium conditions belonging to the spin.

6.2.4 $\mathbf{S}_{nm}^{\cdot\cdot k}$-Condition

Taking $(69)_2$ and the necessary equilibrium condition (52) into account, we obtain from (45) and $(61)_1$

$$\Lambda_{eq}^m \Xi_m^{\cdot k\,eq} = 0. \tag{91}$$

Because the spin stress $\Xi_m^{\cdot k}$ is not regular

$$\Xi_m^{\cdot k} u_k = 0, \qquad u^m \Xi_m^{\cdot k} = 0, \tag{92}$$

according to $(16)_2$, and

$$\Xi_m^{\cdot k\,eq} u_k^{\bullet}{}_{eq} = 0, \qquad u^{m\bullet}{}_{eq} \Xi_m^{\cdot k\,eq} = 0, \tag{93}$$

according to $(61)_1$ and (92), there is the possibility that in equilibrium non-zero temperature-spins are in the kernel of the spin stress as solution of (91).

We obtain from (55) by $(61)_2$ and (70) an other necessary equilibrium condition

$$\left(u^{[n} \Lambda^{m]} S_{nm}^{\cdot\cdot k} \right)_{;k}^{eq} = 0. \tag{94}$$

As derived in appendix 2,

$$u^n \Lambda^m S_{nm}^{\cdot\cdot k} = \frac{1}{2} \Lambda^m \left(\Xi_m^{\cdot k} + \frac{1}{c^2} \Xi_m u^k \right) \tag{95}$$

19

is valid. Consequently, by taking (91), (63) and (61)$_1$ into account, (94) results in

$$0 = \left[\Lambda^m \Xi_m u^k\right]^{eq}_{;k} = (\Lambda^m \Xi_m)^{\bullet}{}_{eq} \tag{96}$$

and according to (43)$_2$, we obtain

$$\Lambda^k \Xi_k = \frac{1}{Tc^2} s_{lm} \Theta^{[lm][ik]} u_i \Xi_k. \tag{97}$$

The spin variables (15), that are the spin density s_{nm} and the spin vector Ξ_m, and the constitutive equations (16), that are the couple stress $s^{\cdot\cdot k}_{nm}$ and the spin stress Ξ^k_m, are related by the spin axioms [13]

$$s_{nm} = \frac{1}{2} \eta_{nmpq} u^p \Xi^q, \tag{98}$$

$$s^{\cdot\cdot k}_{nm} = \frac{1}{2} \eta_{nmpq} u^p \Xi^{qk}. \tag{99}$$

Here η is the Levi-Civita symbol. The spin axioms are caused by the fact that there are only three spin fields and only nine constitutive spin equations [13].

Inserting (98) into (97) results in

$$\Lambda^j \Xi_j = \frac{1}{2Tc^2} \eta^{\cdot\cdot pq}_{lm} \Theta^{[lm][ik]} u_p \Xi_q u_i \Xi_k, \tag{100}$$

and by taking (61)$_1$ into account, (96) becomes

$$0 = \eta^{\cdot\cdot pq}_{lm} \Theta^{[lm][ik]}_{eq} \Xi_q{}^{eq} \Xi_k{}^{eq} (u_p{}^{\bullet}{}_{eq} u_i{}^{eq} + u_p{}^{eq} u_i{}^{\bullet}{}_{eq}). \tag{101}$$

The case of a non-linear coupling tensor is also included, because

$$\Theta^{[lm][ik]\bullet}_{eq}(\Xi_p, \Xi^q_p) = 0 \tag{102}$$

is valid.

Because in (101) the antisymmetric parts of the quadratic forms in (q,k) and (p,i) do not contribute, we obtain

$$[\eta^{\cdot\cdot pq}_{lm} \Theta^{[lm][ik]}_{eq} + \eta^{\cdot\cdot pk}_{lm} \Theta^{[lm][iq]}_{eq} + \eta^{\cdot\cdot iq}_{lm} \Theta^{[lm][pk]}_{eq} + \eta^{\cdot\cdot ik}_{lm} \Theta^{[lm][pq]}_{eq}]$$
$$\Xi_q{}^{eq} \Xi_k{}^{eq} (u_p{}^{\bullet}{}_{eq} u_i{}^{eq} + u_p{}^{eq} u_i{}^{\bullet}{}_{eq}) = 0. \tag{103}$$

20

The tensor of 4th order in the square bracket has the following properties: it is symmetric in (q,k) and in (p,i), and it has an empty kernel according to the coupling property (39). In appendix 3 is proven that the only solutions of (103) are

$$u_p{}^{\bullet}{}_{eq} \neq 0 \qquad \longrightarrow \qquad \Xi_q{}^{eq} = 0 \qquad \longleftrightarrow \qquad \Lambda^q_{eq} = 0, \qquad (104)$$

or

$$\Xi_q{}^{eq} \neq 0 \qquad \longrightarrow \qquad u_p{}^{\bullet}{}_{eq} = 0. \qquad (105)$$

Thus, we proved

> If the acceleration does not vanish in equilibrium, the system has to be spin-free, and if the system is not spin-free, the acceleration has to vanish in equilibrium.

The equilibrium conditions (104) and (105) have to be comparable with (91) and (93). This results in

$$u_p{}^{\bullet}{}_{eq} \neq 0 \qquad \longrightarrow \qquad u_p{}^{\bullet}{}_{eq} \in \ker \Xi^{\cdot p}_{meq} \cap \Xi_p{}^{eq} = 0 \qquad (106)$$
$$\Xi_p{}^{eq} \neq 0 \qquad \longrightarrow \qquad \Xi_p{}^{eq} \in \ker \Xi^{\cdot p}_{meq} \cap u_p{}^{\bullet}{}_{eq} = 0. \qquad (107)$$

We obtain from (103) to (107) that equilibrium is possible in the following cases

$$u_p{}^{\bullet}{}_{eq} = 0 \cap \Xi_q{}^{eq} = 0, \qquad (108)$$
$$u_p{}^{\bullet}{}_{eq} \neq 0 \cap \Xi_q{}^{eq} = 0 \cap u_p{}^{\bullet}{}_{eq} \in \ker \Xi^{\cdot p}_{meq} \qquad (109)$$
$$\Xi_q{}^{eq} \neq 0 \cap u_p{}^{\bullet}{}_{eq} = 0 \cap \Xi_p{}^{eq} \in \ker \Xi^{\cdot p}_{meq}. \qquad (110)$$

As (109) and (110) show, constitutive properties may prevent equilibrium. Whereas in equilibrium the acceleration is always in the kernel of the spin stress according to (92), it depends of the material, if the spin density vector is an element of the kernel of the spin stress in equilibrium. According to (108), the equilibrium is material independent only in spin-free materials with zero acceleration. There are no equilibria with $u_p{}^{\bullet}{}_{eq} \neq 0$ and $\Xi_q{}^{eq} \neq 0$.

7. Recollection

Because thermodynamics and relativity theory in their classical phenomenological versions have a common field of applications, one is challenged to look

for a relativistic continuum thermodynamics. In analogy to non-relativistic continuum thermodynamics, generally one tries to found this theory on the balance equations of energy, momentum, spin and entropy. However, these balances alone do not provide a complete set of conditions to determine all thermodynamic quantities. Thus, like in non-relativistic thermodynamics [19], one has to add constitutive equations describing the material. Therefore it is difficult (or even impossible) to formulate the constitutive equations generally. Insofar, one cannot expect to find a general and complete axiomatic approach to relativistic continuum thermodynamics. The approach which is considered here contains only those conditions which can be formulated without any reference to special classes of materials. The aim of the paper is to present these conditions (especially for equilibrium), well knowing the often overseen fact that finally constitutive equations formulated in a corresponding state space must be supplemented to obtain a closed system of partial differential equations describing a respective class of materials.

According to this program, we start out with the balance equations (1) and (2) which are most general for the following reasons:

1. The balances (1) and (2) are valid in Minkowski space-time as well as in curved space-times which are characterized by a connection defining a covariant derivative (i.e. they are true also in Riemann-, Riemann-Cartan-, and metric-affine space-times),

2. The balances (1) and (2) imply external inputs (the right-hand sides of (1) and (2)), the so-called supplies of energy-momentum, spin, and entropy, and they imply the internal source terms caused by the spin-momentum-energy couplimg (SMEC) depending on the chosen space-time.

3. Entropy supply and entropy production are distiguished in the entropy balance. Therefore, in contrast to other approaches, the dissipation inequality takes the correct form (3) including the supplies.

4. The ansatz (47) for the entropy density 4-vector motivated by the proved identity (21) is most general. It is different from ansatzes in the literature, where the 4-vector of entropy is chosen in such a way that it reflects certain features of non-relativistic relations like the Gibbs-Duhem equation or the Carnot-Clausius relation.

22

In case of a definite theory of gravitation, the connection, and thus the space-time, is specified (e.g. to be Riemann, Einstein-Cartan or metric-affine) and the balance equations are completed by gravitational field equations formulated on the respective space-time. Further, as a consequence of the gravitational field equations and the differential identities valid in the respective type of space-time, the input and SMEC terms in (1) have to be specified, too. Therefore, a (material-independent) theory only based on 1.– 3. is necessarily incomplete in a multiple manner, namely for the missing specification of the state space, the supplies and due to the yet missing specification of the considered gravitational theory. In particular, the equilibrium conditions depend on the gravitational equations, too. For instance, as was shown for GRT [20], most equilibrium conditions adhoc introduced in [11] result from Einstein's equations. Otherwise, the advantage of such an approach considered here is, that it represents a comparatively general framework for possible relativistic continuum thermodynamics.

8. Conclusions

After having proved the unrenouncable entropy identity (21), for the present the most general relativistic expression for the entropy density 4-vector was derived. It contains three parts belonging to particle current, energy-momentum and spin. After that, arguments are given in favor of Eckart's ansatz of the particle flux density 4-vector being parallel to the 4-velocity of the material under consideration. As a consequence, entropy supply and entropy production can be determined as expressions of relativistic invariant terms given by the balances (1) of energy-momentum and spin.

As a further implication of the entropy identity (21) and Eckart's ansatz, it can be shown, that the entropy density 4-vector, ad hoc introduced in [11], is the correct one (in case of the theory of general relativity) except the missing spin part, while the entropy expression given in [14] contradicts the entropy identity (21). The latter follows from the fact that the expression correctly given in [11] must not be supplemented by a time-like vector, as it was supposed in [11] and done in [14].

After the more general considerations, the second part of the paper is devoted to material-independent equilibria in relativistic thermodynamics. For the present, equilibrium is defined by necessary equilibrium conditions: Ac-

cording to the second law, entropy supply, entropy production and entropy 4-flux vanish in equilibrium. From this demand, four equations $((51)_1, (52)$ and $(53))$ follow.

The above mentioned four necessary equilibrium conditions are not sufficient for equilibrium. Consequently, we have to complete these necessary equilibrium conditions by supplementary ones. These *supplementary equilibrium conditions* are

- The vanishing entropy supply results in

  1: the power $(53)_1$ generated by the forces has to vanish in equilibrium. Sufficient for vanishing power is the supplementary equilibrium condition that the forces themselves are zero in equilibrium (58)

  $$u_i{}^{eq} K^i{}_{eq} = 0 \qquad \longleftarrow \qquad K^i{}_{eq} = 0. \qquad (111)$$

  2: if the material is not spin-free, the external moments have to vanish (59). If they do not, the system has to be spin-free

  $$\Lambda^k{}_{eq} \neq 0 \qquad \longrightarrow \qquad L_{[ik]}{}^{eq} = 0, \qquad (112)$$
  $$\Lambda^k{}_{eq} = 0 \qquad \longleftarrow \qquad L_{[ik]}{}^{eq} \neq 0. \qquad (113)$$

- Stemming from the entropy production generated by particle flux density,

  3: the material time derivatives have to vanish in equilibrium, except that of the 4-velocity

  $$\boxplus_m{}^{\bullet eq} := \boxplus_{m;k}{}^{eq} u^k{}_{eq} = 0, \qquad \boxplus_m{}^{eq} \neq u_m. \qquad (114)$$

  4: the expansion (63) has to vanish in equilibrium

  $$u^k_{;k}{}^{eq} = 0. \qquad (115)$$

- Stemming from the entropy production generated by the energy-momentum tensor

5: the heat 4-flux density $(67)_2$ and the entropy 3-flux density $(69)_2$ have to vanish in equilibrium

$$q^k\,_{eq} = 0 \qquad \longrightarrow \qquad \left(\frac{q^k}{T}\right)^{eq} = 0. \qquad (116)$$

6: independently of material and space time, shear and rotation (90) have to be zero in equilibrium

$$\sigma^{eq}_{lk} = 0, \qquad \omega^{eq}_{lk} = 0. \qquad (117)$$

- Stemming from the entropy production generated by the spin tensor

7: equilibrium is possible in the following cases

$$u_p{}^{\bullet}{}_{eq} = 0 \cap \Xi_q{}^{eq} = 0, \qquad\qquad\qquad (118)$$
$$u_p{}^{\bullet}{}_{eq} \neq 0 \cap \Xi_q{}^{eq} = 0 \cap u_p{}^{\bullet}{}_{eq} \in \ker \Xi^{\cdot p}_{m\,eq} \quad (119)$$
$$\Xi_q{}^{eq} \neq 0 \cap u_p{}^{\bullet}{}_{eq} = 0 \cap \Xi_p{}^{eq} \in \ker \Xi^{\cdot p}_{m\,eq}. \quad (120)$$

8: according to (113), external moments need not be zero in equilibrium

9. Appendices

9.1 Appendix 1

Using the projector (8), we obtain

$$N^k = N^l \delta^k_l = N^l(h^k_l + \frac{1}{a^2}u^k u_l) = \frac{1}{a^2}nu^k + n^k. \qquad (121)$$

The same procedure yields

$$T^{ik} = T^{lm}\delta^i_l\delta^k_m = T^{lm}(h^i_l + \frac{1}{a^2}u^i u_l)(h^k_m + \frac{1}{a^2}u^k u_m) =$$
$$= T^{lm}h^i_l h^k_m + \frac{1}{a^2}T^{lm}u^i u_l h^k_m + \frac{1}{a^2}T^{lm}h^i_l u^k u_m + \frac{1}{a^4}T^{lm}u^i u_l u^k u_m =$$
$$= t^{ik} + \frac{1}{a^2}u^i p^k + \frac{1}{a^2}q^i u^k + \frac{1}{a^4}e u^i u^k. \qquad (122)$$

25

9.2 Appendix 2

By use of (11), we obtain

$$
\begin{aligned}
u^n \Lambda^m S_{nm}^{\cdot\cdot k} &= \\
&= u^n \Lambda^m \left[u^k \left(\frac{1}{c^2} s_{nm} + \frac{1}{2c^4} u_n \Xi_m - \frac{1}{2c^4} u_m \Xi_n \right) + \right. \\
&\quad \left. + s_{nm}^{\cdot\cdot k} + \frac{1}{2c^2} u_n \Xi_m^k - \frac{1}{2c^2} u_m \Xi_n^k \right] = \\
&= u^n \Lambda^m u^k \frac{1}{c^2} s_{nm} + u^n \Lambda^m u^k \frac{1}{2c^4} u_n \Xi_m - u^n \Lambda^m u^k \frac{1}{2c^4} u_m \Xi_n + \\
&\quad + u^n \Lambda^m s_{nm}^{\cdot\cdot k} + u^n \Lambda^m \frac{1}{2c^2} u_n \Xi_m^k - u^n \Lambda^m \frac{1}{2c^2} u_m \Xi_n.
\end{aligned}
$$

According to (15) and (16), the terms #1, #3, #4 and #6 are zero. By use of $(7)_1$, we obtain

$$
u^n \Lambda^m S_{nm}^{\cdot\cdot k} = \frac{1}{2c^2} \Lambda^m \Xi_m u^k + \frac{1}{2} \Lambda^m \Xi_m^k \tag{123}
$$

which immediately results in (95).

9.3 Appendix 3

Because the coupling tensor is presupposed as regular, the solutions of (103) are

$$
[\eta_{lm}^{\cdot\cdot pq} \Theta_{eq}^{[lm][ik]} + \eta_{lm}^{\cdot\cdot pk} \Theta_{eq}^{[lm][iq]} + \eta_{lm}^{\cdot\cdot iq} \Theta_{eq}^{[lm][pk]} + \eta_{lm}^{\cdot\cdot ik} \Theta_{eq}^{[lm][pq]}] = 0, \tag{124}
$$

$$
\cup \qquad \qquad \Xi_q^{\;eq} = 0, \tag{125}
$$

$$
\cup \qquad (u_p^{\bullet}{}_{eq} u_i^{\;eq} + u_p^{\;eq} u_i^{\bullet}{}_{eq}) = 0 \quad \longrightarrow \quad u_p^{\bullet}{}_{eq} = 0. \tag{126}
$$

These results are valid for an arbitrary SMEC-term $H_{[ik]}$.

For the present, we investigate (124) in more detail. Multiplication of (124) with arbitrary $A_{(qp)}$, and $B_{(qi)}$ results in

$$
A_{(pq)}[\eta_{lm}^{\cdot\cdot pk} \Theta_{eq}^{[lm][iq]} + \eta_{lm}^{\cdot\cdot iq} \Theta_{eq}^{[lm][pk]}] = 0, \tag{127}
$$

$$
B_{(qi)}[\eta_{lm}^{\cdot\cdot pq} \Theta_{eq}^{[lm][ik]} + \eta_{lm}^{\cdot\cdot ik} \Theta_{eq}^{[lm][pq]}] = 0. \tag{128}
$$

Consequently, we obtain

$$\eta_{lm}^{\cdot\cdot pk}\Theta_{eq}^{[lm][iq]} \;=\; -\eta_{lm}^{\cdot\cdot iq}\Theta_{eq}^{[lm][pk]}, \tag{129}$$

$$\eta_{lm}^{\cdot\cdot pq}\Theta_{eq}^{[lm][ik]} \;=\; -\eta_{lm}^{\cdot\cdot ik}\Theta_{eq}^{[lm][pq]}. \tag{130}$$

For discussing (129), we have to distinguish six different cases

$$
\begin{aligned}
1: \quad & (l,m,p,k) \;\in\; \text{even}\mathcal{P}(1,2,3,4), & (131)\\
2: \quad & (l,m,p,k) \;\in\; \text{odd}\mathcal{P}(1,2,3,4), & (132)\\
3: \quad & (l,m,i,q) \;\in\; \text{even}\mathcal{P}(1,2,3,4), & (133)\\
4: \quad & (l,m,i,q) \;\in\; \text{odd}\mathcal{P}(1,2,3,4), & (134)\\
5: \quad & (l,m,p,k) \;\notin\; \mathcal{P}(1,2,3,4), & (135)\\
6: \quad & (l,m,i,q) \;\notin\; \mathcal{P}(1,2,3,4). & (136)
\end{aligned}
$$

Here $\mathcal{P}(1,2,3,4)$ means a permutation of the elements $(1,2,3,4)$ which can be even or odd. All cases refer to (129).

$$
\begin{aligned}
1\cap 3: \quad & \Theta_{eq}^{[lm][iq]} \;=\; -\Theta_{eq}^{[lm][pk]} \;=\; 0, & (137)\\
1\cap 4: \quad & \Theta_{eq}^{[lm][iq]} \;=\; +\Theta_{eq}^{[lm][pk]} \;=\; 0, & (138)\\
2\cap 3: \quad & -\Theta_{eq}^{[lm][iq]} \;=\; -\Theta_{eq}^{[lm][pk]} \;=\; 0, & (139)\\
2\cap 4: \quad & -\Theta_{eq}^{[lm][iq]} \;=\; +\Theta_{eq}^{[lm][pk]} \;=\; 0, & (140)\\
1\cap 6: \quad & \Theta_{eq}^{[lm][iq]} \;=\; 0, & (141)\\
2\cap 6: \quad & -\Theta_{eq}^{[lm][iq]} \;=\; 0, & (142)\\
3\cap 5: \quad & 0 \;=\; -\Theta_{eq}^{[lm][pk]}, & (143)\\
4\cap 5: \quad & 0 \;=\; +\Theta_{eq}^{[lm][pk]}, & (144)\\
5\cap 6: \quad & 0 \;=\; 0. & (145)
\end{aligned}
$$

A comparison of (131) to (136) with (137) to (144) results in

$$\Theta_{eq}^{[lm][pk]} \;=\; 0, \qquad \text{for all l, m, p, k.} \tag{146}$$

Because the couple tensor $\Theta_{eq}^{[lm][pk]}$ does not vanish identically, we have to dismiss (124), and (125) and (126) remain as possible solutions.

Consequently, we obtain

$$u_p^{\bullet}{}_{eq} \;\neq\; 0 \qquad \longrightarrow \qquad \Xi_q^{eq} \;=\; 0 \qquad \longleftrightarrow \qquad \Lambda_{eq}^q \;=\; 0, \tag{147}$$

27

or

$$\Xi_q{}^{eq} \ne 0 \qquad \longrightarrow \qquad u_p{}^{\bullet}{}_{eq} = 0. \qquad (148)$$

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